GT3

2020/12/03

P1

Theorem ~~3.1~~ III *Let $u_0 \in L^2_{loc}(R^3)$ with $\sup_{x_0\in R^3}\int_{B_1(x_0)}|u_0|^2(x)dx \le \alpha < \infty$. Suppose u_0 is in $L^m(B_2(0))$ with $\|u_0\|_{L^m(B_2(0))} \le M < \infty$ and $m > 3$. Let us decompose*[1] *$u_0 = u_0^1 + u_0^2$ with $\operatorname{div} u_0^1 = 0$, $u_0^1|_{B_{4/3}} = u_0$, $\operatorname{supp} u_0^1 \Subset B_2(0)$ and $\|u_0^1\|_{L^m(R^3)} \le C(M,m)$. Let a be the locally in time defined mild solution to Navier-Stokes equations with initial data u_0^1. Then there exists a positive $T = T(\alpha, m, M) > 0$, such that any Leray solution $u \in \mathcal{N}(u_0)$ satisfies: $u - a \in C^\gamma_{\text{par}}(\overline{B_{1/2}} \times [0,T])$, and $\|u-a\|_{C^\gamma_{\text{par}}(\overline{B_{1/2}}\times[0,T])} \le C(M,m,\alpha)$, for some $\gamma = \gamma(m) \in (0,1)$.*

ε-régularité :

$V\cdot\nabla a$ N-S

$$\begin{cases} \partial_t V - \Delta V + \cancel{a\cdot\nabla V} + \operatorname{div}(a\otimes V) + V\cdot\nabla V + \nabla q = 0 \\ \operatorname{div} V = 0 \end{cases}$$

$\Rightarrow V \in H$

Theorem ~~2.2~~ II (Improved ϵ-regularity criteria) *Let (V, q) be a suitable weak solution to Eq. (2.1) in Q_1, with $a \in L^m(Q_1)$, $\operatorname{div} a = 0$, $\|a\|_{L^m(Q_1)} \le M$, for some $M > 0$ and $m > 5$. Then there exists $\epsilon_1 = \epsilon_1(m, M) > 0$ with the following properties: if*

but.

$$\left(\fint_{Q_1}|V|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1}|q|^{3/2}dxdt\right)^{2/3} \le \epsilon_1,$$

$Q_1 =]-1+t_0, t_0] \times B_1(x_0)$

then V is Hölder continuous in $Q_{1/2}$ with exponent $\alpha = \alpha(m) > 0$ and

scaling.

$$\|V\|_{C^\alpha_{\text{par}}(Q_{1/2})} \le C(m, \epsilon_1, M) = C(m, M). \qquad (2.22)$$

Theorem ~~2.1~~ IV (ϵ-regularity criterion) *Let (V, q) be a suitable weak solution to Eq. (2.1) in Q_1 with $a \in L^m(Q_1)$, $m > 5$, $\operatorname{div} a = 0$. Then there exists $\epsilon_0 = \epsilon_0(m) > 0$ with the following property: if*

$$\left(\fint_{Q_1}|V|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1}|q|^{3/2}dxdt\right)^{2/3} + \left(\fint_{Q_1}|a|^m dxdt\right)^{1/m} \le \epsilon_0, \qquad (2.3)$$

then V is Hölder continuous in $Q_{1/2}$ with exponent $\alpha = \alpha(m) > 0$ and

$$\|V\|_{C^\alpha_{\text{par}}(Q_{1/2})} \le C(m, \epsilon_0). \qquad (2.4)$$

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Theorem ~~2.1~~ IV (ϵ-regularity criterion) *Let* (V, q) *be a suitable weak solution to Eq.* (2.1) *in* Q_1 *with* $a \in L^m(Q_1)$, $m > 5$, $\operatorname{div} a = 0$. *Then there exists* $\epsilon_0 = \epsilon_0(m) > 0$ *with the following property: if*

$$\left(\fint_{Q_1} |V|^3 dxdt\right)^{1/3} + \left(\fint_{Q_1} |q|^{3/2} dxdt\right)^{2/3} + \left(\fint_{Q_1} |a|^m dxdt\right)^{1/m} \leq \epsilon_0, \tag{2.3}$$

then u *is Hölder continuous in* $Q_{1/2}$ *with exponent* $\alpha = \alpha(m) > 0$ *and*

$$\|V\|_{C^{\alpha}_{\text{par}}(Q_{1/2})} \leq C(m, \epsilon_0). \tag{2.4}$$

NS: ε-reg: CKN 82' (Lin 98'

$\alpha <$ { T. Barker, C. Prange }

Stratégie : 1°: Oscillation : (méthod de compacité + Système de Stokes)
linear

2°: Itération :

Notations : $(V)_{Q_R(z_0)} = \dfrac{1}{|Q_R(z_0)|} \displaystyle\int_{Q_R(z_0)} V \, dx\,dt = \fint_{Q_R(z_0)} v \, dx\,dt$

$(v)_{Q_R} = (V)_{Q_R(0)}$

quantité d'oscillation :

$$\mathrm{Osc}(V, q, Q_R(z_0)) = \left(\fint_{Q_R(z_0)} |V - (V)_{Q_R(z_0)}|^3 dz\right)^{1/3} + \left(\fint_{Q_R(z_0)} |q - (q)_{B_R(z_0)}(t)|^{3/2} dz\right)^{2/3} \cdot R.$$

Solution faible suitable : SfS.

$(q)_{B_R(z_0)}(t) \to \fint_{B_R(z_0)} q \, dx.$

P3

$$\begin{cases} \partial_t V - \Delta V + a\cdot\nabla V + \operatorname{div}(a\otimes V) + V\cdot\nabla V + \nabla q = 0 \\ \operatorname{div} V = 0 \end{cases}$$

Lemma d'oscillation :

Soit (V,q) SfS dans Q_1, $a\in L^m(Q_1)$, $m>5$. $\operatorname{div} a = 0$.

$\|a\|_{L^m(Q_1)} \le c$, $\quad |(V)_{Q_1}| \le M \qquad c,\ M>0$.

Alors, pour tout $\theta\in(0,\frac{1}{3})$, il existe $\varepsilon=\varepsilon(\theta,M,m)>0$, $C_1(M,m)>0$.

et $\alpha=\alpha(m)>0$, t.q.

si $\quad \operatorname{Osc}(V,q,Q_1) + |(V)_{Q_1}|\left(\fint_{Q_1}|a|^m\,dx\,dt\right)^{1/m} < \varepsilon \qquad Q_1 \quad < \varepsilon$

alors.

$$\operatorname{Osc}(V,q,Q_\theta) \le C_1(M,m)\,\theta^\alpha\left(\operatorname{Osc}(V,q,Q_1) + |(V)_{Q_1}|\left(\fint_{Q_1}|a|^m\,dx\,dt\right)^{1/m}\right)$$

Q_1

i.e.

$$\boxed{\operatorname{Osc}(V,q,Q_\theta) \le C_1(M,m)\,\theta^\alpha\,\varepsilon} \qquad Q_\theta \subset Q_1$$

$(*)$ $\qquad Q_\theta$

Lemma (Iteration). (V,q), M, $\varepsilon(\theta,M,m)>0$. $C_1(M,m)>0$,

$|(V)_{Q_1}| < M/2$, $\alpha(m)>0$, $a\in L^m(Q_1)$, $\|a\|_{L^m(Q_1)}\le c$,

Soit $\beta=\frac{\alpha}{2}$, on choisit $\theta\in(0,\frac{1}{3})$ t.q. $C_1(M,m)\theta^\alpha \le \theta^\beta$,

$\theta < c_1 = c_1(M,m)$ petit.

Alors il existe $\varepsilon_*(\theta,M,m)$ suffisamment petit. t.q

si $\quad \operatorname{Osc}(V,q,Q_1) + M\left(\fint_{Q_1}|a|^m\,dx\,dt\right)^{1/m} < \varepsilon_*$, (HP)

alors pour $k=1,2,\dots$, on a

- $|(V)_{Q_{\theta^{k-1}}}| \le M$
- $\operatorname{Osc}(V,q,Q_{\theta^{k-1}}) + |(V)_{\theta^{k-1}}|\left(\fint_{Q_{\theta^{k-1}}}|a|^m\right)^{\frac{1}{m}}\theta^{k-1} < \varepsilon_* \le \varepsilon(\theta,M,m)$
- $\operatorname{Osc}(V,q,Q_{\theta^k}) \le \theta^\beta\left(\operatorname{Osc}(V,q,Q_{\theta^{k-1}}) + |(V)_{\theta^{k-1}}|\left(\fint_{Q_{\theta^{k-1}}}|a|^m\right)^{\frac{1}{m}}\theta^{k-1}\right)$

$Q_{\theta^k} \le$

preuve du lemme d'oscillation $\rightarrow \operatorname{div}(V\otimes a)$ $\qquad \operatorname{div}(V\otimes V)$

$$\begin{cases} \underline{\partial_t V - \Delta V} + \underbrace{a\cdot\nabla V + \operatorname{div}(a\otimes V) + V\cdot\nabla V}_{\operatorname{div} F} + \underline{\nabla q} = 0 \\ \operatorname{div} V = 0 \end{cases}$$

Lemme d'oscillation :

Soit (V,q) SfS dans Q_1, $a\in L^m(Q_1)$, $m>5$. $\operatorname{div} a = 0$.
$\|a\|_{L^m(Q_1)} \le c$, $\quad |(V)_{Q_1}| \le M$. $\quad c, M > 0$.
Alors, pour tout $\theta\in(0,\frac13)$, il existe $\varepsilon=\varepsilon(\theta,M,m)>0$, $C_1(M,m)>0$. et $\alpha=\alpha(m)>0$, t.q.

si $\quad Osc(V,q,Q_1) + |(V)_{Q_1}|\left(\fint_{Q_1}|a|^m\,dx\,dt\right)^{1/m} < \varepsilon$

alors.

$$Osc(V,q,Q_\theta) \le C_1(M,m)\,\theta^\alpha\left(Osc(V,q,Q_1) + |(V)_{Q_1}|\left(\fint_{Q_1}|a|^m\,dx\,dt\right)^{1/m}\right)$$

i.e.

$$\underline{Osc(V,q,Q_\theta)} \le \underline{C_1(M,m)\,\theta^\alpha\,\varepsilon}$$

$>$ $\qquad \Rightarrow$ contradiction

La démonstration se fait en 4 étapes.

1° : Considerons le Pb de Stokes suivant :

$$\begin{cases} \partial_t v - \Delta v + \nabla q = \operatorname{div} h \leftarrow \text{tenseur.} \\ \operatorname{div} v = 0 \qquad\qquad \text{ds } Q_1 \end{cases} \qquad \Rightarrow \|v\|_{C^\alpha_{par}(Q_{\frac12})} \le C.$$

2° : Considerons la suite (v_k, q_k), deduire l'éq avec drift.

3° : Compacité

4° : le passage à la limite.

5° : contradiction.

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Étape 1 : Système de Stokes

$$(S)\begin{cases} \partial_t v - \Delta v + \nabla q = \operatorname{div} h = \operatorname{div}(f - a\otimes v - v\otimes a - v\otimes \lambda) \\ \operatorname{div} v = 0 \qquad\qquad \text{ds } Q_1 \end{cases}$$

$\Rightarrow \|v\|_{C^{\alpha}_{par}(Q_{\frac12})} \le C.$

Lemme : Soit $(f_{ij}) = f$, $a \in L^m(Q_1)$. $\left(\int_{Q_1} |f|^m\right)^{1/m} \le M$, $\left(\int_{Q_1} |a|^m\right)^{1/m} \le M$, $|\lambda| \le M$.

Soient $v \in L^\infty L^2 \cap L^2 \dot H^1(Q_1)$ et $q \in L^{3/2}(Q_1)$. et

$$\left(\int_{Q_1} |v|^3\right)^{1/3} + \left(\int_{Q_1} |q|^{3/2}\right)^{2/3} \le M \qquad Q_1$$

(v, q) est une sol distribution de (S).

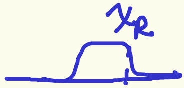
Alors, (v, q) est Hölder continue ds $Q_{1/2}$ et $\|v\|_{C^{\alpha}_{par}(Q_{1/2})} \le C_{M,m}$.

Pf : étape 1 : gain régularité : si $v \in L^p(Q_R)$, $p \ge 3$. $R < 1$.

Δq alors. $v \in L^{\tilde p}(Q_{R-\delta})$. $\tilde p > 1$, $\frac{1}{\tilde p} \ge \frac{1}{p} - \frac{1}{2}\left(\frac{1}{5} - \frac{1}{m}\right)$

$\operatorname{div} \nabla q = \operatorname{div}\operatorname{div} h$

Considerons un cut-off χ_R avec $\frac34 < R < 1$.



décomposons $q := q_1 + q_2$ avec

$$q_1 = \frac{1}{\Delta}\operatorname{div}\operatorname{div}(h\chi_R) \Rightarrow \text{①}\ q_1 = \frac{1}{\Delta}\operatorname{div}\operatorname{div} h \text{ ds } Q_R \rightsquigarrow B_R$$

donc ds Q_R, $\Delta q = \Delta(q_1 + q_2) = \Delta q_1 + \Delta q_2 = \operatorname{div}\operatorname{div} h + \Delta q_2$

ds Q_R. $\Delta q_2 = 0$ ②

pour ① :

$$\|q_1\|_{L^\tau_x(B_R)} = \left\|\frac{1}{\Delta}\operatorname{div}\operatorname{div}(h\chi_R)\right\|_{L^\tau_x(B_R)}$$

$B_R \to \mathbb{R}^3$

....

$$\le \|h\chi_R\|_{L^\tau(\mathbb{R}^3)} \le \|h\|_{L^\tau(B_R)}$$

$$\|q_1\|_{L^\tau_{t,x}(Q_R)} \le \|h\|_{L^\tau_{t,x}(Q_R)} \le M \qquad \tau > \frac32$$

$$\Rightarrow \|q_1\|_{L^{3/2}_{t,x}(Q_R)} \le \|q_1\|_{L^\tau_{t,x}(Q_R)} \le M. \qquad R < 1$$

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pour q_2 : ds Q_R, $\underline{\Delta q_2 = 0}$

Schauder : $\|q_2\|_{C^2_x(B_{R-\frac{\delta}{2}})} \leq C_\delta \|q_2\|_{L^1_x(B_R)}$

$\leq C_\delta \|q - q_1\|_{L^1} \leq \|q-q_1\|_{L^{3/2}}$

$\leq \underbrace{\|q\|_{L^{3/2}}}_{L^{3/2}_t} + \underbrace{\|q_1\|_{L^{3/2}}}_{L^{3/2}_t}$

$\Rightarrow \|q_2\|_{L^{3/2}_t C^2_x(Q_{R-\frac{\delta}{2}})} \leq C(\ q \quad q_2\ )$

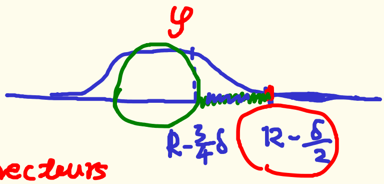
$\leq C_{M,\delta}$

(S) : $\partial_t v - \Delta v + \nabla q = \operatorname{div} h$

$\Rightarrow \underline{(\partial_t - \Delta) v = -\nabla q_1 - \nabla q_2 + \operatorname{div} h}$ ds $Q_{R-\frac{\delta}{2}}$ avec

$\|q_1\|_{L^{\tau}_{tx}(Q_R)} \leq C_M$, $\|q_2\|_{L^{3/2}_t C^2_x(Q_{R-\frac{\delta}{2}})} \leq C_{M,\delta}$, $h \in L^{\tau}_{tx}(Q_R)$ $\tau > \frac{15}{8} > \frac{3}{2}$

En prenant une fonction troncature φ



$(\partial_t - \Delta)(\varphi v) = \varphi D v - [D, \varphi] v$

D

$= \varphi \underbrace{(\partial_t - \Delta) v}_{\text{principale}} - \underbrace{[\partial_t - \Delta, \varphi] v}_{\text{errors} \rightsquigarrow \text{correcteurs}}$

$[\partial_t, \varphi] v = \partial_t \varphi\, v$.

$\varphi = \begin{cases} 1 & Q_{R-\frac{3}{4}\delta} \\ 0, & Q^c_{R-\frac{\delta}{2}} \end{cases}$

$[\Delta, \varphi] v = \Delta(\varphi v) - \varphi \Delta v$

$= \Delta \varphi\, v + 2 \nabla \varphi \nabla v + \cancel{\varphi \Delta v} - \cancel{\varphi \Delta v}$

$Q_{R-\frac{3}{4}\delta}$

ds $Q^c_{R-\frac{3}{4}\delta}$: $\Delta \varphi$, $\nabla \varphi$, $\partial_t \varphi = 0$

$(\partial_t - \Delta)(\varphi v) = \varphi \underline{(\partial_t - \Delta) v} + 0$ ds $Q_{R-\frac{3}{4}\delta}$

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$\underline{\nabla\varphi}\ q_1$
$\quad\ \ {}^{1/2}$

On obtient : $(\partial_t-\Delta)(v\varphi) = \varphi(\partial_t-\Delta)v \ + \ 0$ $\qquad ds \ \underline{Q_{R-\frac{3}{4}\delta}}.$

$= \varphi(-\nabla q_1 - \nabla q_2 + \operatorname{div} h)$

$= \varphi\nabla q_1$

$\nabla\varphi$

$\Rightarrow (\partial_t-\Delta)(v\varphi) = \underline{-\nabla(q_1\varphi)} - \underline{\nabla(q_2\varphi)} + \underline{\operatorname{div}(h\varphi)}$

Duhamel :

$$v\varphi = \underline{e^{t\Delta}(v_0\varphi)} + \underline{\int_0^t e^{(t-s)\Delta}(-\nabla(q_1\varphi) - \nabla(q_2\varphi) + \operatorname{div}(h\varphi))\,ds}$$

$\|q_1\|_{L^{\frac{\tau}{2}}_{tx}(Q_R)} \le C_M, \quad \|h\|_{L^{\tau}_{tx}(Q_R)} \le C.$

$\|q_2\|_{L_t^{\frac{3}{2}} C_x^{2}(Q_{R-\frac{\delta}{2}})} \le C_{M,\delta}$

(1) $\left\| \int_0^t \underline{e^{(t-s)\Delta}}(-\nabla(q_1\varphi) + \operatorname{div}(h\varphi))\,ds \right\|_{L^{r}_{t,x}(Q_{R-\delta})} \le C$, $\quad$ si $\underline{\dfrac{1}{r} > \dfrac{1}{\tau} - \dfrac{1}{5}}$

G_{t-s}

$\|\nabla G_{t-s}\|_{L^p_x} \le C\cdot t^{-(2-\frac{3}{2p})}$

$\left\| \int_0^t G_{t-s}\nabla(h\varphi)\right\|_{L^r_{t,x}(Q_{R-\delta})} \le \left\| \int_0^t \|G_{t-s}\nabla(h\varphi)\|_{L^r_x}\, ds \right\|_{L^r_t}$

$\le \left\| \int_0^t (t-s)^{-(2-\frac{3}{2p})} \|h\varphi\|_{L^\tau}\, ds \right\|_{L^r_t}.$ $\qquad \boxed{1+\frac{1}{r} = \frac{1}{p}+\frac{1}{\tilde\tau}}$

$\frac{1}{|t|^\alpha}$ $\qquad \le \left\| K_\alpha * \|h\varphi\|_{L^{\tilde\tau}_x} \right\|_{L^r}$ $\qquad \alpha = 2-\frac{3}{2p}$

H-L-W
Hardy-L-W : $\ 2-\frac{3}{2p} + \frac{1}{\tilde\tau} = 1 + \frac{1}{r} = \frac{1}{p}+\frac{1}{\tilde\tau}$

$\Rightarrow \quad 2 = \frac{3}{2p}+\frac{1}{p} = \frac{5}{2p} \quad \Rightarrow \quad p = \frac{5}{4}.$ $\ \Rightarrow 1+\frac{1}{r} = \frac{4}{5}+\frac{1}{\tilde\tau}$

$\Rightarrow \frac{1}{r} = \frac{1}{\tilde\tau} - \frac{1}{5}$

$\|h\varphi\|_{L^{\tilde\tau}} \le \|h\|_{L^\tau} \Rightarrow \tilde\tau < \tau$, i.e. $\frac{1}{\tilde\tau} > \frac{1}{\tau}$. $\qquad > \frac{1}{\tau} - \frac{1}{5}$

donc, $\underline{\dfrac{1}{r} > \dfrac{1}{\tau} - \dfrac{1}{5}}.$

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- $\boxed{\|q_2\|_{L^{3/2}_t C^2_x(Q_{R-\frac{\delta}{2}})} \leq C_{\delta,m}.}$

$\Rightarrow \boxed{\left\| \int_0^t e^{(t-s)\Delta} \nabla(q_2 \varphi) \right\|_{L^\infty_{t,x}(Q_{R-\delta})} \leq C_{\delta,m}.}$

En effet.

$$\left| \int_0^t \int_{\mathbb{R}^3} \nabla G_{t-s}(x-y)\, (q_2\varphi)\, dy\, ds \right|$$

$$\leq \int\int \Big| \underbrace{}_{L^1_x} \Big|\, \Big| \underbrace{\nabla(q_2\varphi)}_{L^\infty_x} \Big|\, dy\, ds \qquad W^{1,\infty}$$

$$\leq \int_0^t \underbrace{\|G_{t-s}\|_{L^1_x}}\ \underbrace{\|\nabla(q_2\varphi)\|_{L^\infty_x}}\, ds \leq C \|\nabla(q_2\varphi)\|_{L^1_t L^\infty_x}$$

$$\leq C\|q_2\|_{L^1_t W^{1,\infty}(Q_{R-\delta})} \leq \|q\|_{L^{3/2}_t C^2_x(Q_{R-\delta})}$$

- $\underline{e^{t\Delta}(v_0\varphi)}$ satisfie Heat eq ds $Q_{R-\frac{3}{4}\delta}$ $\Rightarrow$ $e^{t\Delta} v_0\varphi$ smooth in $Q_{R-\delta} \subset Q_{R-\frac{3\delta}{4}}$.

$\Rightarrow$ $v\varphi \in L^r(Q_{R-\delta})$, $\quad \underline{\frac{1}{r} > \frac{1}{t} - \frac{1}{5}} = \frac{1}{p} + \frac{1}{m} - \frac{1}{5}$.

q, h

$\Rightarrow$ $v \in L^r(Q_{R-\delta})$

Donc. on a montré que

$$v \in L^p(Q_R) \Rightarrow v \in L^r(Q_{R-\delta}). \quad \frac{1}{r} > \frac{1}{q} + \underbrace{\frac{1}{m} - \frac{1}{5}}_{\varepsilon}. \qquad \frac{1}{m} - \frac{1}{5} < 0$$

Étape 2°: Itération

On note $\varepsilon = \frac{1}{5} - \frac{1}{m}$.

① $v \in L^p \Rightarrow v \in L^{r_1}(Q_{R-\delta})$. $\frac{1}{r_1} > \frac{1}{q} - \varepsilon$

② $L^{r_1} \Rightarrow v \in L^{r_2}$. $\frac{1}{r_2} > \frac{1}{r_1} - \varepsilon > \frac{1}{q} - 2\varepsilon$

$\vdots\ (Q_{R-n\delta})$

$\Rightarrow v \in L^{r_0}(Q_{\frac{5}{8}})$ $r_0 \gg 1$. $\qquad \frac{3}{4} < R < 1$

On prend r_0 suffisamment grand t.q. $|a||v| \in L^{\bar{r}_0}(Q_{\frac{5}{8}})$, $\bar{r}_0 > 5$

L^m $m > 5$

$\boxed{\text{On a } v \in C^\alpha_{par}(Q_{1/2})}$

$\int_0^t e^{(t-s)\Delta} D g(s,\cdot)\, ds \in C^\alpha_{par}$, si $g \in M^{p,q}$, $q > 5$.

Étape 2 : $\exists$ une suite (v_k, q_k) et (a_k). t.q.

① $\|a_k\|_{L^m(Q_1)} \leq C$, $\mathrm{div}\, a_k = 0$. $|(v_k)_{Q_1}| \leq M$.

② $\mathrm{osc}(v_k, q_k, Q_1) + |(v_k)_{Q_1}| \left(\fint_{Q_1} |a_k|^m \, dx\, dt \right)^{1/m} = \varepsilon_k \to 0$, $k \to +\infty$

③ $\mathrm{osc}(v_k, q_k, Q_\theta) > C_{M,m}\, \theta^\alpha \varepsilon_k$

But : déduire une contradiction.

②: $\left(\fint_{Q_1} |v_k - (v_k)_{Q_1}|^3 dz \right)^{1/3} + \left(\fint_{Q_1} |q_k - (q_k)_{B_1}|^{3/2} dz \right)^{2/3} + |(v_k)_{Q_1}| \left(\fint_{Q_1} |a_k|^m \right)^{\frac{1}{m}} = \varepsilon_k$. $\Downarrow$ 1.

Normalization :

$$\tilde{v}_k = \frac{v_k - (v_k)_{Q_1}}{\varepsilon_k}, \quad \tilde{q}_k = \frac{q_k - (q_k)_{B_1}(t)}{\varepsilon_k}, \quad f_k = \frac{a_k \otimes (v_k)_{Q_1}}{\varepsilon_k}.$$

$$\left(\fint_{Q_1} |\varepsilon_k \tilde{v}_k|^3 dz \right)^{1/3} \qquad \left(\fint_{Q_1} \left(|(v_k)|_{Q_1} |a_k| \right)^m \right)^{1/m}$$

$\Rightarrow$

$$\left(\fint_{Q_1} |\tilde{v}_k|^3 dz \right)^{1/3} + \left(\fint_{Q_1} |\tilde{q}_k|^{3/2} dz \right)^{2/3} + \left(\fint_{Q_1} |f_k|^m dz \right)^{1/m} \overset{\leq C}{=} 1. \checkmark$$

estimation compacte

$$\mathrm{osc}(\tilde{v}_k, \tilde{q}_k, Q_\theta) = \left(\fint_{Q_\theta} |\tilde{v}_k - (\tilde{v}_k)_{Q_\theta}|^3 dz \right)^{1/3} + \theta \cdot \left(\fint_{Q_\theta} |\tilde{q}_k - (\tilde{q}_k)_{B_\theta}|^{3/2} dz \right)^{2/3}$$
$$\geq C_{M,m}\, \theta^\alpha.$$

estimation contradiction.

P10

$$\tilde{V}_k = \frac{V_k - (V_k)_{Q_1}}{\varepsilon_k}$$

Calculer l'éq de $(\tilde{v}_k, \tilde{q}_k)$:

$$\underline{\partial_t V_k - \Delta V_k + a_k \cdot \nabla V_k + \operatorname{div}(a_k \otimes V_k) + V_k \nabla V_k + \nabla q_k = 0}$$

$\partial_t \tilde{v}_k = ----$

$\Delta \tilde{v}_k = ----$

$\varepsilon_k \to 0$

(Eq-P-drift)

$$\begin{cases} \partial_t \tilde{v}_k - \Delta \tilde{v}_k + \underline{a_k \cdot \nabla \tilde{v}_k} + \operatorname{div}(a_k \otimes \tilde{v}_k) + \operatorname{div} f_k + \underbrace{\varepsilon_k \tilde{v}_k \cdot \nabla \tilde{v}_k}_{\to 0} + (v_k)_{Q_1} \cdot \nabla \tilde{v}_k + \nabla \tilde{q}_k = 0 \\ \operatorname{div} \tilde{v}_k = 0 \end{cases}$$

au sens de distribution ds Q_1

$$\Rightarrow \partial_t |\tilde{v}_k| - \Delta |\tilde{v}_k|^2 + 2|\nabla \tilde{v}_k|^2 + \operatorname{div}\left(|\tilde{v}_k|^2 (a_k + \varepsilon \tilde{v}_k + (v_k)_{Q_1})\right)$$
$$+ 2 v_k \operatorname{div}(a_k \otimes \tilde{v}_k + f_k) + 2 \operatorname{div}(\tilde{v}_k \tilde{q}_k) \leq 0$$

au sens de distribution ds Q_1

φ φ

$\nabla \varphi$

$\nabla (v_k \varphi)$

$\nabla v_k \varphi + v_k \otimes \nabla \varphi$

N-S

P11

Étape 3 : Compacité.

$\Rightarrow$

$$\int_{B_1(0)} |\tilde{v}_k(x,t)|^2 \varphi(x)\,dx + 2\int_{-1}^{t}\int_{B_1(0)} |\nabla \tilde{v}_k(x,s)|^2 \varphi(x)\,dx\,ds$$

$0 \leq \varphi \in C_0^\infty((-1\times t] \times B_1(0))$

$$\leq \int_{-1}^{t}\int_{B_1(0)} |\tilde{v}_k|^2(\partial_t\varphi + \Delta\varphi) + (|\tilde{v}_k|^2(a_k + \varepsilon_k \hat{v}_k + (v_k)_{Q_1}))\cdot\nabla\varphi + 2(a\otimes\tilde{v}_k + f_k):(\nabla\tilde{v}_k\varphi + \tilde{v}_k\otimes\nabla\varphi)$$
$$+ 2(\tilde{v}_k\hat{q}_k)\cdot\nabla\varphi \,dx\,dt$$

Def : $A_k(r) = \sup_{-r^2<t<0} \int_{B_r} |\tilde{v}_k(x,t)|^2 dx + 2\int_{-r^2}^{0}\int_{B_r} |\nabla \tilde{v}_k(x,s)|^2 dx\,ds$, pour $0<r<1$.

par l'interpolation, on a déjà $\|\tilde{v}_k\|^2_{L^{10/3}_{t,x}(Q_r)} \leq C A_k(r)$

$L^\infty L^2 \cap L^2 H^1$
$\Downarrow$
$L^{10/3}$

- Δ montrer que $\tilde{v}_k$ est uniforme bornées ds $L^\infty_t L^2 \cap L^2_t \dot{H}^1_x(Q_{3/4})$ ①
- Δ $\tilde{v}_k$ vérifie (Eq - P - drift)

  $\hookrightarrow$ $\partial_t \tilde{v}_k \in L^{3/2}((-(\frac{3}{4})^2, 0); H^{-2}(B_{3/4}))$ ②

par Aubin - Lions - Rellich, on obtient la convergence forte de $\tilde{v}_k$.

① + ② $\Rightarrow$ $\tilde{v}_k$ est précompacte ds $L^{3/2}(Q_{3/4})$

$\Rightarrow$ $\tilde{v}_k \to v$ ds $L^{3/2}(Q_{3/4})$, + $\tilde{v}_k$ bd ds $L^{10/3}(Q_{3/4})$

$\Downarrow$

$\tilde{v}_k \to v$ ds L^r

$\Rightarrow$ $\tilde{v}_k \to v$ in $L^3_{t,x}(Q_{3/4})$]

$r \in (\frac{3}{2}, \frac{10}{3})$

par conséquent.

1. $\tilde{v}_k \to \tilde{v}$ L^3
2. $\tilde{v}_k \to \tilde{v}$ $L^2(B_{3/4})$ pour tout $t \in (-(\frac{3}{4})^2, 0)$
3. $\tilde{q}_k \to \tilde{q}$ $L^{3/2}(Q)$
4. $a_k \to a$ $L^m(Q)$
5. $(v_k)_{Q_1} \to \lambda$ $L^m(Q)$
6. $f_k \to f$ $L^m(Q)$

$f|\tilde{q}_k|^{3/2} \leq 1$